Additional Questions

13. i) *To show convergence is uniform in Question 2.*

Let $a_n \in \mathbb{C}$ be a sequence of coefficients and set $A(x) = \sum_{1 \le n \le x} a_n$. Prove that

$$\sum_{n=N+1}^{\infty} \frac{a_n}{n^s} = -\frac{A(N)}{N^s} + s \int_N^{\infty} \frac{A(t)}{t^{s+1}} dt.$$

ii) Assume there exists a constant $C > 0$ such that $|A(x)| < C$ for all x. Let

$$F(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s} \quad \text{and} \quad F_N(s) = \sum_{n=1}^{N} \frac{a_n}{n^s}.$$

Show that

$$|F(s) - F_N(s)| \le C \frac{1}{N^\sigma} \left(1 + \frac{|s|}{\sigma}\right),$$

for all $N \ge 1$. Thus deduce that for any $\delta > 0$ and $T > 0$ the Dirichlet series for $F(s)$ converges *uniformly* in the semi-infinite rectangle

$$\{s = \sigma + it : \sigma \ge \delta, \, |t| < T\}.$$

Aside This means that the Dirichlet series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^s}$$

of Question 3 is not only convergent but also *holomorphic* for $\operatorname{Re} s > 0$ in which case (50) gives an *analytic* continuation of $\zeta(s)$ to $\operatorname{Re} s > 0$.

14. Generalise Question 3, replacing -1 by any $\zeta \in \mathbb{C}$ satisfying $|\zeta| = 1$ along with $\zeta \ne 1$.

i) Prove that

$$\left| \sum_{n=1}^{N} \zeta^n \right| \le \frac{2}{|1 - \zeta|}$$

for all $N \ge 1$.

ii) Deduce that

$$\sum_{n=1}^{\infty} \frac{\zeta^n}{n^s}$$

converges uniformly for all $\operatorname{Re} s \geq \delta$, for all $\delta > 0$.

iii) Deduce that as long as $\theta \neq 2\pi k$ for any $k \in \mathbb{Z}$, the Dirichlet Series

$$\sum_{n=1}^{\infty} \frac{\sin n\theta}{n^s}$$

converges for all $\operatorname{Re} s > 0$.

15. Generalise Question 11 and show that

$$|\zeta(\sigma+it)| \leq \frac{t^{1-\sigma}}{1-\sigma} + 2t^{1-\sigma} + 1,$$

for $t \geq 4$ and $1/2 \leq \sigma < 1$.

16. For $t > 4$ prove that

$$|\zeta'(1/2+it)| \leq 4t^{1/2}\log t + 4t^{1/2} + 2\log t + 25/8.$$

Hint Either differentiate (29) and (30) or use (32). Then estimate each term. You may estimate some terms differently to how I do in the solution so you may end with different coefficients, though you should be able to get the coefficient of the leading term, $t^{1/2}\log t$, to be 4 as shown.